The Proj construction (Har II 2)

Recall that a graded ring S has a decomposition

$$S = \bigoplus_{d \geq 0} S_d$$

such that if $a_i \in S_i$, $a_j \in S_j$, then $a_i a_j \in S_{i+j}$.

Ex: $k[x_1, \dots, x_n]$ has standard grading $k[x_1, \dots x_n] = \bigoplus_{d \geq 0} A_d$, where A_d is the k-vector space generated by deg d monomials.

If S is any graded ring, $I \subseteq S$ an ideal is homogeneous if it has a homogeneous set of generators (i.e. $\{a_i\}$ where $a_i \in S_{d_i}$, some d_i.)

The ideal $S_+ = \bigoplus_{d > 0} S_d \subseteq S$ is called the irrelevant ideal.
(e.g. in $k[x_1, \dots, x_n]$, the irrelevant ideal consists of polynomials w/ 0 constant term.)

As a set, we define $\operatorname{Proj} S$ to be the set of homogeneous prime ideals not containing S_+.

As a topological space, the closed sets are of the form

$$V(I) := \{ P \in \operatorname{Proj} S \mid P \supset I \}$$

where I is a homogeneous ideal.

For $f \in S$ homogeneous, define

$$D_+(F) := \{ P \in \operatorname{Proj} S \mid F \notin P \} = \operatorname{Proj} S \setminus V(F).$$

Note that $\operatorname{Proj} S \subseteq \operatorname{Spec} S$, and the topology on $\operatorname{Proj} S$ is the one induced by the Zariski topology on Spec. Thus, the sets $D_+(F) = D(F) \cap \operatorname{Proj} S$ form a basis for the topology on $\operatorname{Proj} S$.

$V(-)$ satisfies most of the same properties as in Spec:

- If $I, J \subseteq S$ are homogeneous ideals, then
$$V(IJ) = V(I) \cup V(J).$$

- If $\{I_i\}$ is a family of homogeneous ideals of S then $V(\Sigma I_i) = \bigcap V(I_i)$.

(Proofs of these and most of the rest of the claims in this section are are almost the same as their analogues in the affine case, so we leave them out.)

We define the structure sheaf on $\operatorname{Proj} S$ by defining it on the basis:

Set $\mathcal{O}(D_+(F)) = S_{(F)} := \left\{ \frac{s}{F^t} \;\middle|\; s \in S \text{ is homogeneous and } \deg s = \deg F^t \right\}$

Note that this a ring — a subring of S_F.

If $D_+(F) \subseteq D_+(G)$, the restriction map is the one induced by $S_G \to S_F$.

(We can check that this extends to a sheaf on Proj S by checking the (pre) sheaf axioms on the $D_+(F)$. This is also similar to the proof for Spec.)

Claim: For any $P \in \operatorname{Proj} S$, the stalk $\mathcal{O}_P$ is isomorphic to

$$S_{(P)} := \left\{ \frac{F}{G} \,\middle|\, F, G \in S \text{ homogeneous}, G \notin P, \deg a = \deg u \right\}.$$

Note: $S_{(P)}$ is a local ring with maximal ideal generated by elts of the form $\frac{p}{u}$, where $p \in P$.

Ex: If $S = k[x,y]$ with the standard grading, and $P = (x)$, then $S_{(P)} \cong R$ where R is $k[t]$ localized at the prime ideal (t), via the map

$$\frac{F(x,y)}{G(x,y)} \longmapsto \frac{F(t,1)}{G(t,1)}$$

We now know that Proj S is a locally ringed space, so in order to show it's a scheme, we just need to show that it's covered by affine schemes. In particular, we have the following:

Prop: For each $F \in S$ homogeneous, we have an isomorphism

$$(D_+(F), \mathcal{O}|_{D_+(F)}) \cong \operatorname{Spec} \underbrace{S_{(F)}}_{\text{as defined above}}$$

Pf: We'll give a morphism $(\varphi, \varphi^\#): D_+(F) \to \operatorname{Spec} S_{(F)}$.

First, take the localization map $S \to S_F$. Since $S_{(F)}$ is a subring of S_F, we can define for any $P \in D_+(F)$

$$\varphi(P) = (PS_F) \cap S_{(F)}$$

Since $F \notin P$, PS_F is a prime ideal in S_F, so also in $S_{(F)}$. Moreover, this is a bijection from $D_+(F)$ to $\operatorname{Spec} S_{(F)}$. (Check this!)

To see that φ is a homeomorphism, note that for any homogeneous $I \subseteq S$,

$$P \supseteq I \iff \varphi(P) \supseteq IS_F \cap S_{(F)}.$$

Now if $D(\frac{G}{F^n}) \subseteq \operatorname{Spec} S_{(F)}$ is a basic open set, then we have a natural isomorphism (induced by φ)

$$\mathcal{O}(D(\tfrac{G}{F^n})) \cong S_{(FG)} = \mathcal{O}(D_+(FG)) \quad \text{which}$$

induces an isomorphism

$$\varphi^{\#}: \mathcal{O}_{\operatorname{Spec} S_{(F)}} \longrightarrow \varphi_{*}\left(\mathcal{O}_{\operatorname{Proj} S}\big|_{D_+(F)}\right). \quad \square$$

Cor: Proj S is a scheme.

Def: If A is a ring, define <u>projective n-space over A</u> to be $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$.

Ex: If k is a field we can cover $\mathbb{P}^n_k$ by the open sets $D_+(x_i)$. If $R = k[x_0, \dots, x_n]$, then we have natural isomorphisms

$$R_{(x_i)} \cong k\left[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right] \cong k[X_1, \dots, X_n]$$

so $D_+(x_i) \cong \mathbb{A}^n_k$.

In the case where $k = \bar{k}$, the set of closed points of $\mathbb{P}^n_k$ is homeomorphic to the variety $\mathbb{P}^n$, and the restriction of $\mathcal{O}$ is just the sheaf of regular functions.

Note: The scheme Proj S depends on the grading of S, not just on the ring structure:

Ex: Let $S = k[x, y, z]$ where $\deg x = 2$, $\deg y = \deg z = 1$. Then what is $D_+(x)$?

$$S_{(x)} \cong k\left[\frac{y^2}{x}, \frac{yz}{x}, \frac{z^2}{x}\right] \cong k[A, B, C]/(AC - B^2)$$

so $D_+(x) \cong \operatorname{Spec} k[A,B,C]/(AC-B^2)$ which is not isomorphic to any open set in the standard $\mathbb{P}^2$.

i.e. $\mathbb{P}^2_k \not\cong \operatorname{Proj} S$.